HYDROELASTICITY OF QUARTER-INFINITE PLATE ON WATER OF FINITE DEPTH

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Summary
Diffraction of surface waves by a very large floating platform is studied for the case of finite water depth. In our paper large platform is the flexible thin quarter-infinite plate. By using of ray method we describe the propagation of wave modes inside of the platform area sequentially.

Keywords: diffraction of surface waves, VLFP, deflection, wave modes, ray method, integral-differential equation.

1 Introduction

Recently the problems of the behavior of floating flexible thin plates on waves obtained great interest. This section of hydroelasticity is important and well studied by numerical approaches while remain some difficulties with analytical investigation especially for the case of finite depth.

The thickness of the very large floating platform (VLFP) compared to horizontal parameters is small and they are modeled as thin elastic plates. The dimensions of these objects can be in order of 5km in length and 1km in width while thickness is around 10m.

In HERMANS [1], [2], ANDRIANOV and HERMANS [3], [4] solutions for the deflection and the transmission and reflection coefficients are obtained for infinite, finite and shallow water depth for such forms of the platform: the strip of the infinite length and the semi-infinite plate. The results were presented at the previous IWWWFB in [3]. Here is theory developed further for the quarter-infinite plate.

HERMANS [1] derived an exact integral-differential equation for the deflection of a VLFP at deep water. The equation was solved numerically by means of a boundary element method and a mode expansion. Later HERMANS [2] used this formulation to derive boundary conditions to apply the ray method for short wave diffraction. Some other approaches can be used for the solution of this class of problems, for instance parabolic approximation method, see TAKAGI [5], OHKUSU and NAMBA [6].

Here we study the diffraction of surface waves by VLFP in the form of quarter-infinite plate (QIP) floating on the surface of a fluid of finite depth. It is reasonable to split up the problem on two cases: the first with oblique incident waves and the second with perpendicular waves. We use a ray method for the solution which consists of 3 parts.

The platform is idealized as a plate with elastic properties of zero thickness. In this paper we ignore the effect of corner point. An analytical study is presented for the deflection in all parts of the solution. The first part of the solution is based on the approach ([2], [3]) for the problem of a semi-infinite platform. There an integral-differential equation, a Green's function, boundary and edge conditions are used. One traveling and some evanescent wave modes are considered. A specialty of present approach is in the consideration of the propagation of traveling mode (main ray) of the solution and calculation of its reflection on other edge of the platform (as QIP is considered, platform has 2 edges perpendicular each other). Later a special matching condition introduced along the line which split up the platform area on part where the 'inner' reflection is exist and where is not. This condition is valid in all area of the plate.

2 Formulation of the Problem

We consider a floating flexible thin plate which covers a quarter ($x > 0, y > 0$) of the surface of an ideal incompressible fluid of depth h. z is the positive upwards coordinate. We assume waves in still water and introduce the velocity potential $\nabla\Phi(x,y,z,t) = \vec{V}(x,y,z,t)$ where $\Phi(x,y,z,t)$ is a solution of the Laplace equation

$$\Delta\Phi = 0 \tag{1}$$

in the fluid ($z < 0$) together with the condition at the bottom ($z = -h$) $\partial\Phi/\partial z = 0$ and surface conditions at $z = 0$

$$\frac{\partial\Phi}{\partial z} = \frac{\partial w}{\partial t}, x,y \in \mathcal{P}; \frac{\partial\Phi}{\partial z} = -\frac{1}{g}\frac{\partial^2\Phi}{\partial t^2}, x,y \in \mathcal{F} \tag{2}$$

where $w(x,y,z,t)$ denotes the free surface elevation under the platform, $\mathcal{F}$ is the open fluid area ($-\infty < x < 0, 0 < y < \infty \cup -\infty < x < \infty, -\infty < y < 0$) and $\mathcal{P}$ is the platform area. The dividing surfaces are defined as $\mathcal{S}_y$ ($x = 0, 0 < y < \infty$) and as $\mathcal{S}_x$ ($0 < x < \infty, y = 0$). The fluid region where the incident field and reflected waves from $\mathcal{S}_y$ coexist is defined as $\mathcal{F}_1$ and the region in which incident waves transmitting $\mathcal{S}_x$ as $\mathcal{F}_2$. Incoming short waves propagate from the open fluid in the direction which makes an angle β to x axis (it is shown in figure 1).

The platform draft d is shallow and the platform is assumed to be a thin layer at the free surface. VLFP is modeled then as an elastic plate with zero thickness. To describe the deflection of the platform w we apply the thin plate theory, which leads to a differential equation of the vertical

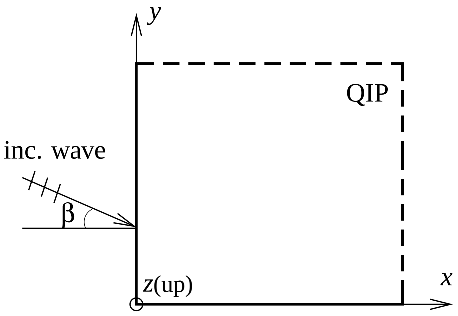

Figure 1: Geometry of the problem

displacement of the platform:

$$D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 w + m\frac{\partial^2 w}{\partial t^2} = P \tag{3}$$

at $z=0$ in the platform area $\mathcal{P}$, where m is the mass of unit area of the platform, D is its equivalent flexural rigidity, $P(x,y,z,t)$ is the linearized pressure

$$P = \rho\frac{\partial\Phi}{\partial t} - \rho g w, \tag{4}$$

here ρ is the density of the water. After applying the operator $\partial/\partial t$ to (3) and using (2) and (4) we arrive at the following equation for Φ at $z=0$:

$$\left\{\mathcal{D}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 + \mu\frac{\partial^2}{\partial t^2}+1\right\}\frac{\partial\Phi}{\partial z}+\frac{1}{g}\frac{\partial^2\Phi}{\partial t^2}=0, \tag{5}$$

where parameters are $\mathcal{D}=D/\rho g$ and $\mu = m/\rho g$.

The edges of the platform are free of moment and shear force, then edge conditions at $\mathcal{S}_y$ $(x=0)$ are:

$$\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}=0;\quad \frac{\partial^3 w}{\partial x^3}+(2-\nu)\frac{\partial^3 w}{\partial x\partial y^2}=0 \tag{6}$$

and at $\mathcal{S}_x$ $(y=0)$:

$$\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}=0;\quad \frac{\partial^3 w}{\partial y^3}+(2-\nu)\frac{\partial^3 w}{\partial y\partial x^2}=0, \tag{7}$$

where ν is Poisson's ratio. The harmonic wave can be written in the form $\Phi(x,y,z,t)=\phi(x,y,z)e^{-i\omega t}$. Then we reduce time-dependence and consider waves of a single frequency ω and obtain at $z=0$:

$$\left(\mathcal{D}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2-\mu+1\right)\frac{\partial\phi}{\partial z}-K\phi=0, \tag{8}$$

where $K=\omega^2/g$. For finite water incident waves equals

$$\phi^{inc}=\frac{\cosh k_0(z+h)}{\cosh k_0 h}\frac{gA}{i\omega}e^{ik_0(x\cos\beta+y\sin\beta)}, \tag{9}$$

here A is the wave height and the wave number k_0 obeys the water dispersion relation $k_0\tanh k_0 h = K$. Length of incoming waves is $\lambda = 2\pi/k_0$.

3 Solution (Platform Area)

We consider the case of oblique waves incoming from the field $(x<0, y>0)$ with angle of incidence β, $0<\beta<\pi/2$. As the effect of corner point $(x=0,y=0)$ is not considered, we may use ray method for this geometry of incident field. An integral-differential formulation derived in HERMANS [2] and ANDRIANOV and HERMANS [4] for the case of finite depth.

The deflection of the platform due to the propagation of main wave modes represented as a superposition of exponential function in the following form

$$w_1(x,y)=\sum_n a_n e^{i\kappa_n x + ik_0 y\sin\beta}, \tag{10}$$

where a_n are the amplitudes of wave modes and κ_n are reduced wave numbers. w_1 is the largest part of the deflection corresponded to the situation when rays transmit $\mathcal{S}_y$ but not reach $\mathcal{S}_x$ yet.

Wave functions κ_n defined as

$$\kappa_n^2 = \kappa^{(n)2} - k_0^2\sin^2\beta, \tag{11}$$

here $\kappa^{(n)}$ are roots of the dispersion relation

$$\left(\mathcal{D}\kappa^4-\mu+1\right)\kappa\tanh\kappa h = k_0. \tag{12}$$

Number $n=N+1$ denotes the number of roots taking into account (one real and N imaginary) lead us to finding $N+3$ unknown amplitudes a_n.

We introduce the Green's function after splitting up the fluid domain and using the thin plate theory for the determination of w_1. If the integral-differential formulation, the surface conditions and Green's theorem are used to the potentials in $\mathcal{P}$ and $\mathcal{F}$ respectively, we obtain such integral-differential equation:

$$\left(\mathcal{D}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2-\mu+1\right)w_1(x,y)=$$
$$\frac{K}{4\pi}\int_{\mathcal{P}}\left(\mathcal{D}\left(\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}\right)^2-\mu\right)w_1(\xi,\eta)\,\mathcal{G}(x,y;\xi,\eta)\,d\xi d\eta$$
$$+Ae^{ik_0(x\cos\beta+y\sin\beta)}, \tag{13}$$

Green's function has the following form:

$$\mathcal{G}(x,y;\xi,\eta)=-2\int_{\mathcal{L}'}\frac{k\cosh kh}{k\sinh kh - K\cosh kh}J_0(kr)\,dk \tag{14}$$

at $z=0$, where $\mathcal{L}'$ is contour of integration in the complex k-plane from 0 to $+\infty$ underneath the singularity $k=k_0$, chosen for fulfilling the radiation condition, $J_0(kr)$ - Bessel's function and $r^2=(x-\xi)^2+(y-\eta)^2$.

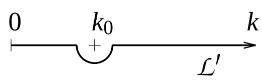

Figure 2: Contour of integration

We consider the zeros of dispersion relation for water surface and then replacing of (10) into (13) result in

$$\sum_n a_n\mathcal{K}\frac{\left(\mathcal{D}\kappa^{(n)^4}-\mu\right)}{(\kappa_n-k_0\cos\beta)\cos\beta}+A=0. \tag{15}$$

Here $\mathcal{K} = k_0 K / \left(K(1-Kh)+k_0^2 h\right)$. To complete the system of $N+3$ equations, two equations may be obtained from edge conditions (6). Now amplitude functions a_n and function w_1 can be received.

We investigate the propagation of main ray further. After the reflection of traveling wave mode (main ray) on $\mathcal{S}_x$ with an angle θ (that is shown in figure 3) the QIP get the added deflection w_2 generated by the vibration of the plate edge $\mathcal{S}_x$:

$$w_2(x,y) = \sum_n a_n^* e^{i\kappa_n^* y + i\kappa_1 x}, \qquad (16)$$

where a_n^* are the amplitudes after the reflection at $\mathcal{S}_x$ and corresponded wave numbers are $\kappa_n^{*2} = \kappa^{(n)2} - \kappa_1^2$. By this way $\kappa_1^* = -k_0 \sin\beta$ and the largest term of reflected traveling wave is $a_1^* e^{-ik_0 y \sin\beta + i\kappa_1 x}$. In mathematical plan a_n^* are the solutions of the system depending from $a_1(\kappa_1)$ and consisting from integral-differential and edge equations at $y = 0$.

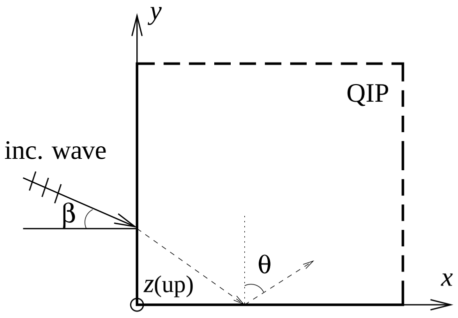

Figure 3: Reflection of traveling wave mode (main ray) at x-axis

In general, the deflection of the QIP can be written in the following form:

$$w(x,y) = w_1(x,y) + w_2(x,y) \qquad (17)$$

where $w_1(x,y)$ and $w_2(x,y)$ were described already. The reflected part of the deflection w_2 exists in the region $x/\tan\theta > y$. The angle of the ray reflection θ may be obtained from $\tan\theta = k_0/\kappa_1$.

To find w_2 we apply an analysis similar to the determination of w_1. Integral-differential formulation leads us to the equation for a_n^*:

$$\sum_n a_n^* \mathcal{K} \frac{\left(\mathcal{D}\kappa^{(n)^4} - \mu\right)}{(\kappa_n - k_0\cos\theta)\cos\theta} + a_1(\kappa_1) = 0 \qquad (18)$$

with the two edge conditions (7).

Now deflection functions w_1 and w_2 are known. We obtain the following intermediate results for total deflection $w(x,y)$ in platform zones: in $\mathcal{P}_1$ ($x < y\tan\theta$) $w = w_1$ and in $\mathcal{P}_2$ ($x > y\tan\theta$) $w = w_1 + w_2$.

In next section the present approach will be improved by the introducing a special matching condition along the border between these zones and we will find a new part of total deflection.

4 Matching Condition

Here we derive the matching conditions along the border $x = y\tan\theta$ which split up the platform area in $\mathcal{P}_1$ and $\mathcal{P}_2$. By introducing of these conditions, the construction of the solution for whole area of the platform will be completed. Instead of the function w_2 will used new deflection function, which exist in both zones.

For the case of oblique waves we use a 'straight-forward' ray method. For a perpendicular incident field OHKUSU and NAMBA [6] obtained a solution for platform which floats on shallow water by using a parabolic approximation and matching of different zones.

We use the general formulation for the diffraction of waves by QIP and apply the ray method. The solution of the Laplace equation (1) is $\Phi(\vec{x},t)$ and the potential of undisturbed incident wave ϕ^{inc} is given by (9).

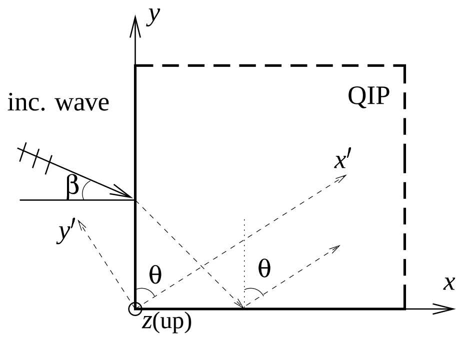

Figure 4: Geometry of current analysis

Let us consider the new system of coordinates $0x'y'$ which is obtained by rotation angle $\pi/2 - \theta$ from $0xy$ and x' is the border between $\mathcal{P}_1$ and $\mathcal{P}_2$ and its direction is coinciding with direction of rays after reflection at $y = 0$.

We assume that the potential underneath the platform can be written as a superposition of ray-mode solutions

$$\phi(x) = \sum_n \varphi_n(x) e^{i\kappa x + \kappa z}, \qquad (19)$$

where $\varphi_n(x)$ is the amplitude function of the nth mode. We skip the primes. The deflection is represented by

$$w_s(x,y) = \frac{iA}{\omega g}\phi_z(x,y). \qquad (20)$$

We assume that our approximation is valid in whole area of the platform. Insertion of (19) to Laplace equation (1) gives:

$$\Delta\varphi + 2\kappa(i\varphi_x + \varphi_z) + O(K^0) = 0. \qquad (21)$$

We write $y^* = K^{1/2}y$, $\kappa = Kr$ and eliminate the z-derivative by the following expression

$$\varphi_z = -i\varphi_x - \frac{\varphi_{y^*y^*}}{2r}. \qquad (22)$$

Insertion of (19) into (8) lead us after dropping the asterisk to:

$$O(K): \quad \left(\mathcal{D}r^4 - \mu + 1\right)r = 1. \qquad (23)$$

and

$$O(K^0): \quad \left(\mathcal{D}r^4 - \mu + 1\right)\left(-i\varphi_x - \frac{\varphi_{yy}}{2r}\right) + \mathcal{D}\left(-2r^3\varphi_{yy} - 4ir^4\varphi_x\right) = 0 \qquad (24)$$

(23) is the dispersion relation at $z = 0$. Now partial differential equation

$$2ir\varphi_x + \varphi_{yy} = 0 \tag{25}$$

have to be considered. Two terms are equal in order of magnitude when $x = O(1)$ and $y = O(K^{1/2})$.

Use of Laplace transform $\psi(s,y) = \int_0^\infty \varphi(x,y)e^{-sx}dx$ lead us to the following equation:

$$2irs\psi + \psi_{yy} = 2ir\varphi(0,y) \tag{26}$$

and we require the initial conditions, see MEI and TUCK [7], which differs with change of sign of y:

$$\varphi(0,y) = A_s,\ y < 0; \tag{27}$$

$$\varphi(0,y) = 0,\ y > 0. \tag{28}$$

Value of the constant A_s may be obtained from the second part of our solution by using of the results for w_2.

From general solution of (26) we arrive to the equations:

$$\psi^-(s,y) = \alpha_1(s)e^{\sqrt{2irsy}} + \frac{A_s}{s}, \tag{29}$$

$$\psi^+(s,y) = \beta_2(s)e^{-\sqrt{2irsy}}. \tag{30}$$

with special notation of the amplitude function and its transforms: φ^- and ψ^- in region $y < 0$ and φ^+ and ψ^+ in region $y > 0$. We can determine the constants: $\alpha_1 = -\beta_2 = -A_s/2s$. Applying of Laplace inverse transform and using of transition conditions $\varphi^+ = \varphi^-$ and $\varphi^+_y = \varphi^-_y$ at $y = 0$ lead us to the following formulas for the amplitude function:

$$\varphi^-(x,y) = -\frac{A_s}{\sqrt{\pi}}\left(\int_0^\infty e^{-\lambda^2}d\lambda - \int_0^{\frac{y\sqrt{2ir}}{2\sqrt{x}}} e^{-\lambda^2}d\lambda\right) + A_s, \tag{31}$$

$$\varphi^+(x,y) = \frac{A_s}{\sqrt{\pi}}\left(\int_0^\infty e^{-\lambda^2}d\lambda - \int_0^{\frac{y\sqrt{2ir}}{2\sqrt{x}}} e^{-\lambda^2}d\lambda\right). \tag{32}$$

After the calculations of the integral values of the potential in the platform area and respectively of the deflection w_s may be determined too.

5 Results & Conclusions

Finally, the total deflection of the quarter-infinite platform is written as the following sum:

$$w(x,y) = w_1 + w_s \tag{33}$$

in the whole area of the platform, where the first term is the main part of the solution and the second part represents the solution along the rays by stretching the coordinates.

This approach and analysis of the behavior of VLFP will be developed further. The details and obtained results will be presented and discussed at the Workshop.

Later we will extend the presented approach to the case when incident waves propagate perpendicularly to the platform and to the case when incident waves propagate from $(-\infty, -\infty)$. The same analysis we are going to extend to the problem for two other forms of the platform: the strip of semi-infinite length and plate of finite sizes. The latter is a more realistic case and it is the main goal of our study.

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Question by : T. Miloh
Is it true that your solution is valid only away from the origin and is it possible to obtain a local solution at the corner?

Author's reply:
Yes, indeed our solution is not valid near the corner of the plate.
Yes, it seems possible to obtain a solution at the corner point and in shadow region generated by the presence of the corner. Hopefully, on the next Workshop we will show these results.

Question by : M. Ohkuhu
A problem you treated with is almost the same as those studied in the papers of Takagi (2002) and Ohkusu (2003) you referred. It seems though more general, is somehow mathematically related to the ones of those papers. I wonder if you would give a description of the relationship of them.

Author's reply:
Yes, in all three papers the quarter-infinite plate is considered. But in our paper, the general case - case of finite depth is studied. Then, by taking the limits, we can solve the problems for infinite and shallow water, which is considered in the referred papers.
The results we presented are valid for restricted values of the angle of incidence. In future we will extend our approach to unrestricted incident angle.

Question by : D.V Evans
Is your theory a ray theory, only valid for short waves?

Author's reply:
In the derivation of the direct and reflected wave fields we have not used small values of the wavelength explicitly. However, to obtain smooth solution we introduced a coordinate stretching to end up with a parabolic equation. So our approach is valid only for short waves.
